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2003 J. Phys. A: Math. Gen. 36 5663

(http://iopscience.iop.org/0305-4470/36/20/321)

INSTITUTE OF PHYSICS PUBLISHING JOURNAL OF PHYSICS A: MATHEMATICAL AND GENERAL

J. Phys. A: Math. Gen. **36** (2003) 5663–5681 PII: S0305-4470(03)57159-1

The stationary Maxwell–Dirac equations

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Received 4 December 2002, in final form 24 March 2003
Published 7 May 2003
Online at stacks.iop.org/JPhysA/36/5663

Abstract

The Maxwell–Dirac equations are the equations for electronic matter, the 'classical' theory underlying QED. The system combines the Dirac equations with the Maxwell equations sourced by the Dirac current. A stationary Maxwell–Dirac system has $\psi = \mathrm{e}^{-\mathrm{i}Et}\phi$, with ϕ independent of t. The system is said to be isolated if the dependent variables obey quite weak regularity and decay conditions. In this paper, we prove the following strong localization result for isolated, stationary Maxwell–Dirac systems,

- there are no embedded eigenvalues in the essential spectrum, i.e. $-m \leqslant E \leqslant m$;
- if $|E| < m$ then the Dirac field decays exponentially as $|x| \to \infty$;
- if $|E| = m$ then the system is 'asymptotically' static and decays exponentially if the total charge is non-zero.

PACS numbers: 11.10.Lm, 11.10.−z, 12.90.+b, 13.40.−f

1. Introduction

The Maxwell–Dirac system consists of the Dirac equation

$$\gamma^\alpha(\partial_\alpha - \mathrm{i}eA_\alpha)\psi + \mathrm{i}m\psi = 0 \tag{1}$$

with electromagnetic interaction given by the potential A_α; and the Maxwell equations (sourced by the Dirac current, j^α),

$$F_{\alpha\beta} = \partial_\alpha A_\beta - \partial_\beta A_\alpha \qquad \partial^\alpha F_{\alpha\beta} = -4\pi e j_\beta = -4\pi e\bar{\psi}\gamma_\beta\psi. \tag{2}$$

Most studies of the Dirac equation treat the electromagnetic field as given and ignore the Dirac current as a source for the Maxwell equations, i.e. these treatments ignore the electron 'self-field'. A comprehensive survey of these results can be found in the book by Thaller [1]. This is not surprising, inclusion of the electron self-field via the Dirac current leads to a very difficult, highly non-linear set of partial differential equations. So difficult in fact that the existence theory and solution of the Cauchy problem was not completed until 1997—seventy years after Dirac first wrote down his equation! In a stunning piece of non-linear analysis, worked out over almost a twenty year period, Flato *et al* [2] solved the Cauchy problem for small initial data. Other contributors to this work on the existence of solutions would include Gross [3], Chadam [4], Georgiev [5], Esteban *et al* [6] and Bournaveas [7].

There are no known non-trivial, explicit solutions, i.e. exact solutions in terms of elementary functions, to the Maxwell–Dirac equations in 1 + 3 dimensions—all known solutions involve some numerical work. These solutions do, however, exhibit interesting non-linear behaviour which would not have been apparent through perturbation expansions. The particular solutions found in [8–10] exhibit just this sort of behaviour—localization and charge screening. See also Das [11] and the more recent work of Finster *et al* [12].

Finster *et al* also point out in [13] that solving the system (Einstein–Maxwell–Dirac system in their case) gives, in effect, all the Feynman diagrams of the quantum field theory, with the exception of the Fermionic loop diagrams. Study of the Maxwell–Dirac system should provide an interesting insight into non-perturbative QED.

The aim of the present work is to obtain qualitative information on stationary solutions of the Maxwell–Dirac system, in doing so one would hope to be able to say something about Maxwell–Dirac models of some simple natural systems involving the electron. In fact, it is surprising that so little is understood about these systems. There is no Maxwell–Dirac model for a single isolated electron. If we compare the situation to that in the other great physical theory of the 20th century, General Relativity, the situation could not be more stark. There are a host of solutions to Einstein's equations representing single, isolated gravitating bodies.

In this paper we will prove the following result:

Main theorem. *A stationary, isolated Maxwell–Dirac system has no embedded eigenvalues, i.e.* $-m \leqslant E \leqslant m$.

If $|E| < m$ *then the Dirac field, decays exponentially as* $|x| \to \infty$.
If $|E| = m$ *then the system is 'asymptotically static' and, if the system has non-vanishing total charge, decays exponentially as* $|x| \to \infty$.

The content of the theorem will be established through theorems 1 to 5 in sections 4 to 7. Parts of the theorem (e.g., no embedded eigenvalues) are reasonably well known under somewhat different assumptions—see for instance the book by Thaller [1] and the work of Berthier and Georgescu [14]—although the approach taken here is rather different in that the fields are subject to only weak regularity and decay conditions in the asymptotic spatial region. This approach means that we find an explicit expression for E in terms of the limiting values of the field variables.

The paper is organized as follows. First, we give a brief overview of 2-spinor methods applied to the Maxwell–Dirac equations (some details of 2-spinor calculus may also be found in the appendices). In section 3 the definitions of stationary, isolated systems are given and some simple consequences are explored. In section 4 we address the important question of embedded eigenvalues in the spectrum of the Dirac operator, the main results being proposition 1 and theorem 1. In section 5 we examine regularity and decay issues with the main results given in theorems 2 and 3. The next section, section 6, looks at the special case $|E| = m$ and its relation to a generalization of the static systems of [16]. In section 7 we prove the exponential decay of the Dirac field in the $|E| = m$ case. Finally, in section 8, we conclude with a brief discussion of the results.

2. The Maxwell–Dirac Equations

In this section, we give a very brief account of the 2-spinor formulation of the Maxwell–Dirac equations, details may be found in [9], some of the 'mechanics' of the 2-spinor formalism are collected in appendix A at the end of the paper.

In [9], the 2-spinor form of the Dirac equations was employed to solve (1) for the electromagnetic potential, under the non-degeneracy condition $j^\alpha j_\alpha \neq 0$. In terms of 2-spinors—see below—the non-degeneracy condition can be written as $u_C v^C \neq 0$, since $j^\alpha j_\alpha = 2|u_C v^C|^2$. Requiring A^α to be a real four-vector gives a set of partial differential equations in the Dirac field alone, *the reality conditions*.

For 2-spinors u_A and v^B (see [15] for an exposition of the 2-spinor formalism), we have

$$\psi = \begin{pmatrix} u_A \\ \bar{v}^{\dot{B}} \end{pmatrix} \qquad \text{with} \quad u_C v^C \neq 0 \text{ (non-degeneracy)}$$

where $A, B = 0, 1$, $\dot{A}, \dot{B} = \dot{0}, \dot{1}$ are 2-spinor indices. The Dirac equations are

$$(\partial^{A\dot{A}} - \mathrm{i}eA^{A\dot{A}})u_A + \frac{\mathrm{i}m}{\sqrt{2}}\bar{v}^{\dot{A}} = 0 \qquad (\partial^{A\dot{A}} + \mathrm{i}eA^{A\dot{A}})v_A + \frac{\mathrm{i}m}{\sqrt{2}}\bar{u}^{\dot{A}} = 0 \tag{3}$$

where $\partial^{A\dot{A}} \equiv \sigma^{\alpha A\dot{A}}\partial_\alpha$ and $A^{A\dot{A}} = \sigma^{\alpha A\dot{A}}A_\alpha$; here $\sigma^{\alpha A\dot{A}}$ are the Infeld–van der Waerden symbols.

The electromagnetic potential is (see [9] for details),

$$A^{A\dot{A}} = \frac{\mathrm{i}}{e(u^c v_c)}\left\{v^A\partial^{B\dot{A}}u_B + u^A\partial^{B\dot{A}}v_B + \frac{\mathrm{i}m}{\sqrt{2}}(u^A\bar{u}^{\dot{A}} + v^A\bar{v}^{\dot{A}})\right\}. \tag{4}$$

The reality conditions are,

$$\begin{aligned}
&\partial^{A\dot{A}}(u_A\bar{u}_{\dot{A}}) = -\frac{\mathrm{i}m}{\sqrt{2}}(u^C v_C - \bar{u}^{\dot{C}}\bar{v}_{\dot{C}}) \\
&\partial^{A\dot{A}}(v_A\bar{v}_{\dot{A}}) = \frac{\mathrm{i}m}{\sqrt{2}}(u^C v_C - \bar{u}^{\dot{C}}\bar{v}_{\dot{C}}) \\
&u_A\partial^{A\dot{A}}\bar{v}_{\dot{A}} - \bar{v}_{\dot{A}}\partial^{A\dot{A}}u_A = 0.
\end{aligned} \tag{5}$$

The Maxwell equations are,

$$\partial^\alpha F_{\alpha\beta} = -4\pi e j_\beta = -4\pi e\sqrt{2}\sigma_\beta^{A\dot{A}}(u_A\bar{u}_{\dot{A}} + v_A\bar{v}_{\dot{A}}). \tag{6}$$

The equations (4), (5) and (6) are entirely equivalent to the original Maxwell–Dirac equations, (1) and (2).

3. Isolated, stationary Maxwell–Dirac systems

We recall the definitions of [16] for stationary and isolated systems.

Definition 1. *A Maxwell–Dirac system is said to be stationary if there is a gauge in which* $\psi = \mathrm{e}^{\mathrm{i}\omega t}\phi$, *with the bi-spinor* ϕ *independent of t. Such a gauge will be referred to as a stationary gauge.*

Clearly, a stationary gauge is not unique—any gauge transformation $\psi \to \mathrm{e}^{\mathrm{i}\omega t}\psi$ leaves the system in a stationary gauge. Note that under such a gauge change we have, $A^\alpha \to A^\alpha - \frac{\omega}{e}\delta_0^\alpha$. We are interested in isolated systems, i.e. systems for which the fields decay suitably as $|x| \to \infty$, in this case we will require that $A^\alpha \to 0$ as $|x| \to \infty$ in some stationary gauge. In this particular gauge we will write $\psi = \mathrm{e}^{-\mathrm{i}Et}\phi$, for the stationary gauge in which $A^0 \to 0$ as $|x| \to \infty$. Note that for any stationary system in a stationary gauge A^α is independent of time, t, (see equation (4)).

In most physical processes that we would wish to model using the Maxwell–Dirac system, we would be interested in isolated systems—systems where the fields and sources are largely confined to a compact region of $\mathbb{R}^3$. This requires that the fields decay sufficiently quickly as $|x| \to \infty$.

The best language for the discussion of such decay conditions and other regularity issues is the language of weighted function spaces: specifically weighted classical and Sobolev spaces. In [16] the weighted Sobolev spaces, $W^{k,p}_\delta$, were used following the definitions of [17]. These definitions have the advantage that the decay rate is explicit: under appropriate circumstances a function in $W^{k,p}_\delta$ behaves as $|x|^\delta$ for large $|x|$. An element, f, of $W^{k,p}_\delta$ has $\sigma^{-\delta+|\alpha|-\frac{3}{p}}\partial^{|\alpha|}f$ in L^p for each multi-index α for which $0 \leqslant |\alpha| \leqslant k$; here $\sigma = \sqrt{1+|x|^2}$ and we are working on $\mathbb{R}^3$ (or some appropriate subset thereof)—see [17] or [18] and [19] (the latter papers use a different indexing of the Sobolev spaces). We will make use of the Sobolev inequality and frequent use of the multiplication lemma.

Sobolev inequality (see [17, 18]). *If $f \in W^{k,p}_\delta$ then*

(i)

$$\|f\|_{\frac{np}{(n-kp)},\delta} \leqslant C\|f\|_{k,q,\delta} \qquad \textit{if} \quad n-kp>0 \quad \textit{and} \quad p \leqslant q \leqslant \frac{np}{(n-p)}$$

(ii)

$$\|f\|_{\infty,\delta} \leqslant C\|f\|_{k,p,\delta} \qquad \textit{if} \quad n-kp<0 \quad \textit{and} \quad |f(x)| = o(r^\delta) \qquad \textit{as} \quad r \to \infty.$$

Multiplication lemma (see [18]). *Pointwise multiplication on E_ρ is a continuous bilinear mapping*

$$W^{k_1,2}_{\delta_1} \times W^{k_2,2}_{\delta_2} \to W^{k,2}_\delta$$

if $k_1, k_2 \geqslant k$, $k < k_1 + k_2 - n/2$ and $\delta > \delta_1 + \delta_2$.

We will be interested in the asymptotic region (spatially) of the Maxwell–Dirac system, which we denote by $E_\rho = \mathbb{R}^3 \backslash B_\rho$, where B_ρ is the ball of radius ρ. We will take our fields to be elements of the function spaces $W^{k,p}_\delta(E_\rho)$ for certain values of the indices k, p and δ. Before introducing the precise definition of an isolated system, we must (following [16]) introduce some notation.

Suppose we have a stationary system and we are in a stationary gauge for which $A^\alpha \to 0$ as $|x| \to \infty$. Write, $u_A = \mathrm{e}^{-\mathrm{i}Et}U_A$ and $\bar{v}^{\dot{A}} = \mathrm{e}^{-\mathrm{i}Et}\bar{V}^{\dot{A}}$ with U_A, V_A and A^α all independent of time, t. Note that $u_C v^C = U_C V^C$ is a gauge and Lorentz invariant complex scalar function, this means we can introduce a (unique up to sign) 'spinor dyad' $\{o_A, \iota_B\}$ with $\iota^A o_A = 1$—some facts on 2-spinor dyads are collected in appendix A at the end of the paper. The dyad is defined as follows, let $U_C V^C = R\,\mathrm{e}^{\mathrm{i}\chi}$—where R and χ are real functions—then write,

$$U_A = \sqrt{R}\,\mathrm{e}^{\mathrm{i}\frac{\chi}{2}}o_A \qquad \text{and} \qquad V_A = \sqrt{R}\,\mathrm{e}^{\mathrm{i}\frac{\chi}{2}}\iota_A.$$

Note that we must have $R > 0$ (almost everywhere) because of our non-degeneracy condition.

We can now define our isolated systems, note this definition is a little more general than the definition of [16].

Definition 2. *A stationary Maxwell–Dirac system will be said to be isolated if, in some stationary gauge, we have*

$$\psi = \mathrm{e}^{-\mathrm{i}Et}\sqrt{R}\begin{pmatrix}\mathrm{e}^{\frac{\mathrm{i}\chi}{2}}o_A\\ \mathrm{e}^{-\frac{\mathrm{i}\chi}{2}}\bar{\iota}^{\dot{A}}\end{pmatrix}$$

with E constant and $\sqrt{R} \in W^{3,2}_{-\tau}(E_\rho)$; $\mathrm{e}^{\frac{\mathrm{i}}{2}\chi}o_A$, $\mathrm{e}^{\frac{\mathrm{i}}{2}\chi}\iota_A \in W^{3,2}_{\epsilon}(E_\rho)$ and $A^\alpha \in W^{2,2}_{-1+\epsilon}(E_\rho)$, for some $\tau > \frac{3}{2}$, $\rho > 0$ and any $\epsilon > 0$.

Remarks.

- This definition ensures, after use of the Sobolev inequality and the multiplication lemma, that $\psi = o(r^{-\tau+\epsilon})$ and $A^\alpha = o(r^{-1+\epsilon})$.
- Note our condition places regularity restrictions on the fields in the region E_ρ only. In the interior of B_ρ there are no regularity assumptions.
- A minimal condition that one may impose on the Dirac field is that it have finite total charge in the region E_ρ, this amounts to
$$\begin{aligned}\int_{E_\rho} j^0\,\mathrm{d}x &= \int_{E_\rho}(|U_0|^2 + |U_1|^2 + |V^0|^2 + |V^1|^2)\,\mathrm{d}x\\ &= \int_{E_\rho} R(|o_0|^2 + |o_1|^2 + |\iota^0|^2 + |\iota^1|^2)\,\mathrm{d}x < \infty.\end{aligned}$$
This, of course, simply means that U_A and V^A are in $L^2(E_\rho)$. So U_A and V^A would have L^2 decay at infinity; roughly, they would decay faster than $|x|^{-\frac{3}{2}}$, i.e. we require at least $\tau > \frac{3}{2}$.
- The spherically symmetric solution of [9] provides an excellent example of an *isolated*, *stationary* and *static* Maxwell–Dirac system.

With our assumption that the Maxwell–Dirac system is isolated and stationary, we can impose the Lorenz gauge condition, without altering the above regularity and decay assumptions on A^α. To see this we note that

$$\partial_\alpha A^\alpha = \sum_{j=1}^{3}\partial_j A^j \in W^{1,2}_{-2+\epsilon}(E_\rho)$$

and that the Laplacian, Δ, gives an isomorphism $W^{3,2}_{\epsilon} \to W^{1,2}_{-2+\epsilon}$ (we may assume $\epsilon < 1$)—see [20, 21] (and also [17–19]). This means there is a unique solution, $\Omega \in W^{3,2}_{\epsilon}(E_\rho)$, of the equation

$$\Delta\Omega + \partial_\alpha A^\alpha = 0.$$

Consequently, for the gauge change $A_\alpha \to \hat{A}_\alpha = A_\alpha + \partial_\alpha\Omega$ we still have $\hat{A}^\alpha \in W^{2,2}_{-1+\epsilon}$, with $\hat{A}^\alpha$ satisfying the Lorenz gauge condition.

The electromagnetic potential of our stationary, isolated Maxwell–Dirac system will be taken to satisfy the two equations,

$$\Delta A^\alpha = 4\pi e\sqrt{2}\sigma^{\alpha A\dot{A}}R(o_A\bar{o}_{\dot{A}} + \iota_A\iota_{\dot{A}}) \tag{7}$$

$$\partial_\alpha A^\alpha = \sum_{j=1}^{3} \frac{\partial A^j}{\partial x^j} = 0. \tag{8}$$

To end this section we present a simple result which we will need in the following sections.

Lemma 1. *For a stationary and isolated Maxwell–Dirac system, in the Lorenz gauge,*

$$A^0 - \frac{q_0}{|x|} \in W^{5,2}_{-\eta}(E_\rho) \qquad q_0 \text{ a constant} \quad \text{and} \tag{9}$$

$$A^j \in W^{5,2}_{-\eta}(E_\rho) \quad j = 1, 2, 3 \qquad \text{and} \qquad \eta = 2(\tau - 1) > 1. \tag{10}$$

Proof. Firstly we note that the source term of the Maxwell equation (7) is in $W^{3,2}_{-2\tau+2\epsilon}(E_\rho)$, i.e. $j^\alpha \in W^{3,2}_{-2\tau+2\epsilon}(E_\rho)$. The Laplacian gives an isomorphism between $W^{5,2}_{-2(\tau-\epsilon-1)}$ and $W^{5,2}_{-2\tau+2\epsilon}$ see [20, 21] (also [17–19]). So there exists an $a^\alpha \in W^{5,2}_{-2(\tau-\epsilon-1)} = W^{5,2}_{-\eta}$ such that,

$$\Delta a^\alpha = 4\pi e j^\alpha.$$

Now, as $A^\alpha \in W^{2,2}_{-1+\epsilon}(E_\rho)$, we have

$$A^\alpha = \frac{q_\alpha}{|x|} + a^\alpha,$$

where the q_α are constants. Applying the Lorenz condition we find $q_j = 0$, for $j = 1, 2, 3$. □

We can improve the decay rates here to higher (negative) order harmonic polynomials, at the expense of regularity, by using the Dirac equations to get $j^\alpha \in W^{2,2}_{-2(\tau+1)+2\epsilon}(E_\rho)$—but lemma 1 is sufficient for our purposes.

The constant q_0 is the total electric charge of the system (i.e., the electric charge of the Dirac field plus the charge due to any external sources in B_ρ); this is easily seen by taking a Gauss integral over the sphere at infinity of the electrostatic field (given by the gradient of A^0).

4. No Embedded eigenvalues

A famous theorem of H Weyl asserts the invariance of the essential spectrum of the perturbation of an operator if the difference of the resolvents of the perturbed and original operators is compact, see section 4.3.4 of [1]. In standard notation we have, for a stationary system with $\psi = \mathrm{e}^{-\mathrm{i}Et}\phi$,

$$H\phi = E\phi \qquad \text{with} \qquad H = \gamma^0 \sum_{j=1}^{3} \gamma^j \left(-\mathrm{i}\frac{\partial}{\partial x^j} + eA^j \right) + (\gamma^0 m - eA^0)$$

the free operator, H_0, has $A^\alpha = 0$. Consequently, we have

Proposition 1. *The Dirac Hamiltonian operator H of a stationary and isolated Maxwell–Dirac system has the same essential spectrum as the free operator, i.e.*

$$\sigma_{ess}(H) = \sigma_{ess}(H_0) = \sigma(H_0) = (-\infty, -m] \cup [m, \infty).$$

Proof. Note that H satisfies the conditions of section 4.3.4 in [1], because of our decay assumptions on the electromagnetic potential. So the theorem follows as a simple adaption of the theorem of [1]. □

We will now prove that for our stationary isolated systems there are no embedded eigenvalues, E (the 'energy').

Theorem 1. *A stationary and isolated Maxwell–Dirac system has no embedded eigenvalues E, i.e. $-m \leqslant E \leqslant m$. In particular, the following limit exists,*

$$\frac{E}{m} = \lim_{|x|\to\infty} \frac{\cos\chi}{\sqrt{1+\frac{1}{2}\lambda^2}} \qquad \text{where} \quad \lambda^2 = \sum_{j=1}^{3}(l^j + n^j)^2$$
$$\text{with} \quad l^\alpha = \sigma^\alpha_{A\dot{A}} o^A \bar{o}^{\dot{A}} \quad \text{and} \quad n^\alpha = \sigma^\alpha_{A\dot{A}} \iota^A \bar{\iota}^{\dot{A}}.$$

Remarks.

- The result needs the Maxwell equations only in order to derive the decay result for A^α of lemma 1. The result remains true for the Dirac equation alone if we assume the appropriate decay for A^α.
- The only conditions required are the rather weak regularity and decay conditions of an isolated system. No positivity conditions on the potential are required, cf section 4.7.2 of [1].

Proof. The proof is remarkably simple, it is a matter of exploiting the notation introduced in definition 2. We begin by re-writing the 2-spinor form of the Dirac equations, (3), in this notation. We have,

$$\begin{aligned} &\frac{o_A}{2}\left(\frac{\partial^{A\dot{A}}R}{R} + \mathrm{i}\partial^{A\dot{A}}\chi\right) + \partial^{A\dot{A}}o_A - \mathrm{i}eB^{A\dot{A}}o_A + \frac{\mathrm{i}m}{\sqrt{2}}\,\mathrm{e}^{-\mathrm{i}\chi}\bar{\iota}^{\dot{A}} = 0 \\ &\frac{\iota_A}{2}\left(\frac{\partial^{A\dot{A}}R}{R} + \mathrm{i}\partial^{A\dot{A}}\chi\right) + \partial^{A\dot{A}}\iota_A + \mathrm{i}eB^{A\dot{A}}\iota_A + \frac{\mathrm{i}m}{\sqrt{2}}\,\mathrm{e}^{-\mathrm{i}\chi}\bar{o}^{\dot{A}} = 0 \end{aligned} \tag{11}$$

where $B^\alpha = E\delta^\alpha_0 + A^\alpha$.

We combine these two equations into a single (equivalent) equation which gives the derivative $\frac{\partial_\alpha R}{R} + \mathrm{i}\partial_\alpha\chi$. To do this multiply the first equation by ι_B and the second by o_B and subtract. Using $o_A\iota_B - o_B\iota_A = \epsilon_{AB}$ (see appendix A), we have

$$\begin{aligned} \frac{\partial_{A\dot{A}}R}{R} + \mathrm{i}\partial_{A\dot{A}}\chi - 2\mathrm{i}(\iota_A o_B + \iota_B o_A)B^B{}_{\dot{A}} + 2\big(\iota_A\partial^B{}_{\dot{A}}o_B - o_A\partial^B{}_{\dot{A}}\iota_B\big) \\ + \sqrt{2}\mathrm{i}m(\iota_A\bar{\iota}_{\dot{A}} - o_A\bar{o}_{\dot{A}})\,\mathrm{e}^{-\mathrm{i}\chi} = 0. \end{aligned} \tag{12}$$

We now use the multiplication lemma to place the terms of the equation into an appropriate weighted Sobolev space:

- $\partial_{A\dot{A}}\chi \in W^{2,2}_{-1+2\epsilon}(E_\rho)$, since (from definition 2),
$$\mathrm{e}^{\mathrm{i}\chi} = \mathrm{e}^{\mathrm{i}\frac{\chi}{2}}\iota^A\,\mathrm{e}^{\mathrm{i}\frac{\chi}{2}}o_A \in W^{3,2}_{2\epsilon}(E_\rho);$$
- $(\iota_A o_B + \iota_B o_A)B^B{}_{\dot{A}} = (\iota_A o_B + \iota_B o_A)\big(E\sigma^{0B}{}_{\dot{A}} + A^B{}_{\dot{A}}\big)$, breaking this into two terms we have,
$$(\iota_A o_B + \iota_B o_A)E\sigma^{0B}{}_{\dot{A}} \in W^{3,2}_{2\epsilon}(E_\rho) \qquad (\iota_A o_B + \iota_B o_A)A^B{}_{\dot{A}} \in W^{2,2}_{-1+2\epsilon}(E_\rho)$$
this last inclusion uses lemma 1;
- $\iota_A\partial^B{}_{\dot{A}}o_B - o_A\partial^B{}_{\dot{A}}\iota_B \in W^{2,2}_{-1+2\epsilon}(E_\rho)$.

So that equation (12) implies,

$$\frac{\partial_{A\dot{A}}R}{R} - 2\mathrm{i}(\iota_A o_B + \iota_B o_A)E\sigma^{0B}{}_{\dot{A}} + \sqrt{2}\mathrm{i}m(\iota_A\iota_{\dot{A}} - o_A\bar{o}_{\dot{A}})\,\mathrm{e}^{-\mathrm{i}\chi} \in W^{2,2}_{-1+2\epsilon}(E_\rho). \tag{13}$$

We now contract equation (13) with $o^A\bar{o}^{\dot{A}}$ and so on. Using, the results of appendix A (in particular the last two facts), we have, after splitting the resulting equations into real and imaginary parts,

$$\frac{\partial_\alpha R}{R} + \sqrt{2}m\sin\chi(n_\alpha - l_\alpha) + \mathrm{i}(m^0 - \bar{m}^0)E(m^0\bar{m}_\alpha - \bar{m}^o m_\alpha) \in W^{2,2}_{-1+4\epsilon}(E_\rho) \tag{14}$$

$$(l^0 - n^0),\ (m^0 + \bar{m}^0)E,\ (l^0 + n^0)E - \sqrt{2}m\cos\chi \in W^{2,2}_{-1+4\epsilon}(E_\rho). \tag{15}$$

Now (using appendix A),

$$l^0 + n^0 = \frac{1}{\sqrt{2}}\left(|o_0|^2 + |o_1|^2 + |\iota_0|^2 + |\iota_1|^2\right) > 0$$

and

$$(l^\alpha + n^\alpha)\eta_{\alpha\beta}(l^\beta + n^\beta) = (l^0 + n^0)^2 - \sum_{j=1}^{3}(l^j + n^j)^2 = 2$$

here $\eta_{\alpha\beta}$ is the Minkowski metric. So we have,

$$\frac{1}{\sqrt{2}}(l^0 + n^0) = \frac{1}{\sqrt{2}}(l_0 + n_0) = \sqrt{1 + \frac{1}{2}\sum_{j=1}^{3}(l^j + n^j)^2} = \sqrt{1 + \frac{1}{2}\lambda^2}.$$

From the last inclusion of equation (15) we have

$$\frac{1}{\sqrt{2}}(l^0 + n^0)E - m\cos\chi = \sqrt{1 + \frac{1}{2}\lambda^2}E - m\cos\chi \in W^{2,2}_{-1+4\epsilon}(E_\rho).$$

Note that, $\left(1 + \frac{1}{2}\lambda^2\right)^{-\frac{1}{2}} \in W^{3,2}_{2\epsilon}$ and consequently,

$$\frac{E}{m} - \frac{\cos\chi}{\sqrt{1 + \frac{1}{2}\lambda^2}} \in W^{2,2}_{-1+6\epsilon}(E_\rho).$$

Finally, from the Sobolev inequality we have,

$$\left|\frac{E}{m} - \frac{\cos\chi}{\sqrt{1 + \frac{1}{2}\lambda^2}}\right| < C|x|^{-1+6\epsilon}$$

for any $\epsilon > 0$ and some constant C. Hence the limit $|x| \to \infty$ of the left-hand side exists and is zero, which completes the proof. □

5. Regularity and decay

A stationary Maxwell–Dirac system is an elliptic system of partial differential equations. So it should be a simple matter to apply the theory of elliptic regularity to obtain the best possible regularity results for the Maxwell and Dirac fields. That this is indeed the case is demonstrated in the next theorem.

Theorem 2. *A stationary Maxwell–Dirac system for which, $U_A, V_A \in L^2(E_\rho)$ and $A^\alpha \in L^1_{loc}(E_\rho)$, is C^∞, i.e. U, V and A are in $C^\infty(E_\rho)$.*

Proof. First we note that we can always find a gauge transformation which takes A into the Lorenz gauge while leaving it in the same Lebesgue space.

We have a set of Poisson equations for the A (equation (7)), and the elliptic Klein–Gordon equations (B.2) (i.e., the second-order equations derived from the first-order Dirac equations). These equations take the form,

$$\Delta A^\alpha = 4\pi e j^\alpha \qquad \Delta U_A + 2\mathrm{i}e\sum_{j=1}^{3} A^j\partial_j U_A = (m^2 - E^2)U_A + h_A^B U_B$$

where h_A^B is a quadratic function of the A; with a similar equation for the V. As U and V are in $L^2(E\rho)$, j^α is in $L^1(E_\rho)$, so the A must be in $L^3_{loc}(E_\rho)$. Putting this information into the U and V equations, we conclude (after the use of the Hölder inequality) that the $h_B^A U_A$ are in $L^{\frac{6}{5}}_{loc}(E_\rho)$ and so the U and V are in $L^3_{loc}(E_\rho)$ (elliptic regularity). We can now conclude that all the fields are in $C^{0,\alpha}$ and then in $C^{2,\alpha}$. Iterating, we finally find that the fields are in $C^\infty(E_\rho)$ (this is the classical 'bootstrap' argument, see ch 10 of [23] or ch 9 of [22]). □

In the case of the free Klein–Gordon equation it is easy to see that, in the stationary case, if $E^2 - m^2 < 0$ then the Dirac field decays exponentially. This fact remains true for our stationary isolated systems.

Theorem 3. *For an isolated and stationary Maxwell–Dirac system, with $E^2 \neq m^2$, the Dirac fields U and V (along with all their derivatives) decay exponentially as $|x| \to \infty$.*

Proof. With $E^2 \neq m^2$ and theorem 1 we have $|E| < m$.

We now cast our Klein–Gordon equations in a form suitable for the application of the maximum principle. Firstly, $\Delta(|U|^2) = \bar{U}\Delta U + U\Delta\bar{U} + 2|\nabla U|^2$, so using equation (B.3):

$$\begin{aligned}\Delta(|U_0|^2 + |U_1|^2) = {} & 2\mathrm{i}e\sum_{j=1}^{3}(U_0A^j\partial_j\bar{U}_0 - \bar{U}_0A^j\partial_jU_0U_1A^j\partial_j\bar{U}_1 - \bar{U}_1A^j\partial_jU_1)\\ & + 2(m^2 - E^2)(|U_0|^2 + |U_1|^2) + 2(|\nabla U_0|^2 + |\nabla U_1|^2)\\ & - 2[2eEA^0 + e^2A^\alpha A_\alpha](|U_0|^2 + |U_1|^2) - \mathrm{i}e(\bar{\partial}A - \partial\bar{A})(|U_0|^2 - |U_1|^2)\\ & - 2\mathrm{i}e[(-\partial_zA + \partial A^3)U_0\bar{U}_1 - (-\partial_z\bar{A} + \bar{\partial}A^3)\bar{U}_0U_1]. \end{aligned} \qquad (16)$$

We will use the following comparison function,

$$w(x) = C_0\frac{\mathrm{e}^{-\sqrt{2}k|x|}}{|x|} \quad \text{with} \quad C_0 = \rho\,\mathrm{e}^{\sqrt{2}k\rho}\sup_{|x|=\rho}[h(x)] \qquad \text{and} \qquad k > 0,$$

where $h(x) = (|U_0|^2 + |U_1|^2 + |V_0|^2 + |V_1|^2)$; the supremum in the definition of C_0 is well defined and finite since we have by the Sobolev inequality, $0 < h(x) < C|x|^{-2(\tau-\epsilon)}$, on E_ρ, for ρ large enough. Note that we have $\Delta w - 2k^2w = 0$, on E_ρ.

We show that

$$\Delta[h(x) - w(x)] - 2k^2[h(x) - w(x)] \geqslant 0.$$

The following inequalities will be needed—in each case we have used the Sobolev inequality on the A (after use of lemma 1). Firstly,

$$\mathrm{i}\sum_{j=1}^{3}(U_0A^j\partial_j\bar{U}_0 - \bar{U}_0A^j\partial_jU_0U_1A^j\partial_j\bar{U}_1 - \bar{U}_1A^j\partial_jU_1)$$

$$\begin{aligned}
&= R\sum_{j=1}^{3}[A^j\partial_j\chi(|o_0|^2+|o_1|^2) + \mathrm{i}(o_0A^j\partial_j\bar{o}_0 \\
&\quad - \bar{o}_0A^j\partial_jo_0 + o_1A^j\partial_j\bar{o}_1 - \bar{o}_1A^j\partial_jo_1)] \\
&\quad > -2\frac{C_1}{|x|}R \\
&\quad > -\frac{C_1}{|x|}R(|o_0|^2+|o_1|^2+|\iota_0|^2+|\iota_1|^2) \\
&= -\frac{C_1}{|x|}(|U_0|^2+|U_1|^2+|V_0|^2+|V_1|^2)
\end{aligned}$$

where we have used,

$$1 = |\iota^Ao_A| = |\iota^0o_0+\iota^1o_1| \leqslant \tfrac{1}{2}(|o_0|^2+|o_1|^2+|\iota_0|^2+|\iota_1|^2).$$

Next, in a similar vein, we have

$$-2[2eEA^0+e^2A^\alpha A_\alpha] > -\frac{C_2}{|x|}$$

$$\begin{aligned}
-\mathrm{i}e(\bar{\partial}A-\partial\bar{A})(|U_0|^2-|U_1|^2) &> -e|\bar{\partial}A-\partial\bar{A}|(|U_0|^2+|U_1|^2) \\
&> -\frac{C_3}{|x|^2}(|U_0|^2+|U_1|^2)
\end{aligned}$$

$$\begin{aligned}
-2\mathrm{i}e[(-\partial_zA+\partial A^3)U_0\bar{U}_1 - (-\partial_z\bar{A}+\bar{\partial}A^3)\bar{U}_0U_1] &> -2\frac{C_4}{|x|^2}|U_0U_1| \\
&> -\frac{C_4}{|x|^2}(|U_0|^2+|U_1|^2)
\end{aligned}$$

here the C_j are positive constants.

With the use of equation (16) and the inequalities we have,

$$\begin{aligned}
\Delta[(|U_0|^2+|U_1|^2)] \geqslant{}& \left[2(m^2-E^2) - \frac{C_2}{|x|} - \frac{C_3}{|x|^2} - \frac{C_4}{|x|^2} - \right](|U_0|^2+|U_1|^2) \\
&- \frac{C_1}{|x|}h(x) + 2(|\nabla U_0|^2+|\nabla U_1|^2).
\end{aligned}$$

There is, of course, an entirely similar equation for $|V^0|^2+|V^1|^2$. Adding these two equations gives, for every k such that $0<k<\sqrt{m^2-E^2}$,

$$\begin{aligned}
&\Delta[h(x)-w(x)] - 2k^2[h(x)-w(x)] \\
&\quad \geqslant \left[2(m^2-E^2-k^2) - \frac{C}{|x|}\right]h(x) + (|\nabla U_0|^2+|\nabla U_1|^2+|\nabla V_0|^2+|\nabla V_1|^2) \\
&\quad \geqslant 0, \text{ for } \rho \text{ large enough.}
\end{aligned}$$

Applying the maximum principle on E_ρ we see that the non-negative maximum of $h(x)-w(x)$ must occur at infinity or on $|x|=\rho$. However,

$$\lim_{|x|\to\infty}[h(x)-w(x)] = 0 \qquad \text{and} \qquad [h(x)-w(x)]_{|x|=\rho} \leqslant 0.$$

We conclude that $[h(x)-w(x)] \leqslant 0$ on E_ρ, so that

$$|U_0|^2 + |U_1|^2 + |V^0|^2 + |V^1|^2 \leqslant C_0 \frac{e^{-\sqrt{2}k|x|}}{|x|}.$$

So the U and V decay exponentially.

Differentiating the Klein–Gordon equations we can use the same procedure to show that the first derivatives decay exponentially. After taking account of theorem 2, we can iterate this procedure once we note that the solution for A^α can be written as the sum of an harmonic polynomial (of negative degree) and the convolution of j^α and the appropriate Green's function. □

This theorem does not deal with decay of solutions in the case $|E| = m$. It is clear that the solutions in this case need not decay exponentially. The spherically symmetric solution of [9] is a case in point, this solution has $|E| = m$ and $R \sim \frac{C_0}{|x|^4}$ as $|x| \to \infty$.

The case $|E| = m$ is in a sense quite unique, as we will see in the next section.

6. $|E| = m$ and asymptotically static systems

In this section, we will prove a theorem which neatly ties together the $|E| = m$ case and the concept of an asymptotically static solution.

In [16] the idea of a static Maxwell–Dirac system was exploited to show that if the system was also stationary and isolated then the system was necessarily electrically neutral, with $|E| = m$. A static Maxwell–Dirac system is one for which (in some Lorentz frame) the spatial components of the Dirac current vanish. With the Dirac current written as,

$$j^\alpha = \sqrt{2}\sigma^\alpha{}_{A\dot{A}}(u^A\bar{u}^{\dot{A}} + v^A\bar{v}^{\dot{A}}) = \sqrt{2}R(l^\alpha + n^\alpha) = \sqrt{2}R\sigma^\alpha{}_{A\dot{A}}(o^A\bar{o}^{\dot{A}} + \iota^A\bar{\iota}^{\dot{A}})$$

we would require for a static system that $l^k + n^k = 0, k = 1, 2, 3$. We will now generalize this concept to that of an *asymptotically static* Maxwell–Dirac system.

Definition 3. *A Maxwell–Dirac system will be called asymptotically static, with decay rate* κ *and differentiability index s, if* $l^k + n^k = \sigma^k{}_{A\dot{A}}(o^A\bar{o}^{\dot{A}} + \iota^A\bar{\iota}^{\dot{A}}) \in W^{s,2}_{-\kappa}(E_\rho)$ *for some* $\rho, \kappa > 0$ *and* $k = 1, 2, 3$.

So an asymptotically static system decays towards a static system, as $|x| \to \infty$. The unit vector, $\frac{1}{\sqrt{2}}(l^\alpha + n^\alpha)$ (in the direction of the current, j^α) has only a time-like component in the limit as $|x| \to \infty$.

To take full advantage of this definition, we will need to recast it in terms of the individual variables o_A, and ι_A, this is done in the following lemma.

Lemma 2. *An isolated and stationary Maxwell–Dirac system is asymptotically static if and only if*

$$\iota^0 - \bar{o}_{\dot{0}} \qquad \iota^1 - \bar{o}_{\dot{1}} \in W^{t,2}_{-\kappa+\epsilon}(E_\rho)$$

where $t = \min[3, s]$.

Proof. First assume the system is asymptotically static with decay rate κ and differentiability s.

From the proof of theorem 1 and definition 3, we have

$$\begin{aligned}(|o_0|^2 + |o_1|^2 + |\iota^0|^2 + |\iota^1|^2)^2 - 4 &= 2(l^0 + n^0)^2 - 4 \\ &= 2\sum_{k=1}^{3}(l^k + n^k)^2 \in W^{s,2}_{-2\kappa}(E_\rho).\end{aligned}$$

Consequently, as $[(|o_0|^2 + |o_1|^2 + |\iota^0|^2 + |\iota^1|^2) + 2]^{-1} \in W^{3,2}_{2\epsilon}(E_\rho)$, we have

$$|o_0|^2 + |o_1|^2 + |\iota^0|^2 + |\iota^1|^2 - 2 \in W^{t,2}_{-2\kappa+2\epsilon}(E_\rho).$$

Writing out the $l^k + n^k$ explicitly we have,

$$-(o_0\bar{o}_{\dot{1}} + o_1\bar{o}_{\dot{0}}) + (\iota^0\bar{\iota}^{\dot{1}} + \iota^1\bar{\iota}^{\dot{0}}) \in W^{s,2}_{-\kappa}(E_\rho)$$
$$-(-o_0\bar{o}_{\dot{1}} + o_1\bar{o}_{\dot{0}}) + (-\iota^0\bar{\iota}^{\dot{1}} + \iota^1\bar{\iota}^{\dot{0}}) \in W^{s,2}_{-\kappa}(E_\rho)$$
$$-|o_0|^2 + |o_1|^2 + |\iota^0|^2 - |\iota^1|^2 \in W^{s,2}_{-\kappa}(E_\rho).$$

So we conclude that,

$$-o_0\bar{o}_{\dot{1}} + \iota^1\bar{\iota}^{\dot{0}} \in W^{s,2}_{-\kappa}(E_\rho) \qquad \text{and} \qquad |o_0|^2 + |\iota^1|^2 - 1, |o_1|^2 + |\iota^0|^2 - 1 \in W^{t,2}_{-\kappa}(E_\rho).$$

Writing these equations as a single matrix equation,

$$\begin{pmatrix} o_1 & -\iota^0 \\ o_0 & \iota^1 \end{pmatrix}\begin{pmatrix} \bar{o}_{\dot{0}} - \iota^0 & \bar{o}_{\dot{1}} - \iota^1 \\ \bar{\iota}^{\dot{1}} - o_1 & o_0 - \bar{\iota}^{\dot{0}} \end{pmatrix} = \begin{pmatrix} o_1\bar{o}_{\dot{0}} - \iota^0\bar{\iota}^{\dot{1}} & |o_1|^2 + |\iota^0|^2 - 1 \\ |o_0|^2 + |\iota^1|^2 - 1 & o_0\bar{o}_{\dot{1}} - \iota^1\bar{\iota}^{\dot{0}} \end{pmatrix}$$

is in $W^{t,2}_{-\kappa}(E_\rho)$. The first matrix on the left has determinant 1 and inverse,

$$\begin{pmatrix} o_1 & -\iota^0 \\ o_0 & \iota^1 \end{pmatrix}^{-1} = \begin{pmatrix} \iota^1 & \iota^0 \\ -o_0 & o_1 \end{pmatrix}$$

which is in $W^{3,2}_{\epsilon}(E_\rho)$. The result now follows from the multiplication lemma after applying this inverse matrix to the previous equation.

Next, assuming $\iota^0 - \bar{o}_{\dot{0}}, \iota^1 - \bar{o}_{\dot{1}} \in W^{t,2}_{-\kappa+\epsilon}(E_\rho)$, we easily find that $l^k + n^k \in W^{t,2}_{-\kappa+2\epsilon}(E_\rho)$, $k = 1, 2, 3$, and any $\epsilon > 0$. So the system is asymptotically static. □

Now to our theorem connecting the two apparently unrelated notions, the condition $|E| = m$ and the idea of an asymptotically static system.

Theorem 4. *A stationary and isolated Maxwell–Dirac system is asymptotically static if* $|E| = m$.

Proof. Assume the system is stationary and isolated with $|E| = m$; write $E = \varepsilon m$, with $\varepsilon = \pm 1$.

The proof is very simple, it simply involves manipulating expressions obtained in the proof of theorem 1. From that proof we have,

$$\sqrt{1 + \tfrac{1}{2}\lambda^2} - \varepsilon \cos\chi \in W^{2,2}_{-1+4\epsilon}(E_\rho).$$

Note that we must have $\varepsilon \cos\chi \geqslant 0$, for ρ large enough, so

$$\sqrt{1 + \tfrac{1}{2}\lambda^2} + \varepsilon \cos\chi \in W^{2,2}_{4\epsilon}(E_\rho).$$

Multiplying the last two expressions and using the multiplication lemma, we have

$$\sin^2\chi + \tfrac{1}{2}\lambda^2 \in W^{2,2}_{-1+8\epsilon}(E_\rho).$$

These equations together with the fact that both $\sin^2\chi$ and $l^k + n^k$ are in $W^{3,2}_{2\epsilon}(E_\rho)$ enable us to conclude that

$$\sin\chi, l^k + n^k \in W^{2,2}_{-\frac{1}{2}+4\epsilon}(E_\rho).$$

The system is, according to definition 3, asymptotically static. □

7. Exponential decay, the $|E| = m$ case

In this section we will prove that the Dirac field decays exponentially in the $|E| = m$ case as well—at least when the total charge $q_0 \neq 0$. In fact we obtain tight bounds on the decay of the Dirac field in this case.

Theorem 5. *The Dirac field for a stationary, isolated and asymptotically static Maxwell–Dirac system, with $|E| = m, \kappa > 1$, $s = 3$ (definition 3) and $q_0 \neq 0$ decays exponentially as $|x| \to \infty$. In fact, there exists two positive constants C_1 and C_2 such that*

$$C_1 \frac{\mathrm{e}^{-4\sqrt{2}m\lambda\sqrt{|x|}}}{|x|^{\frac{3}{2}}} < R < C_2 \frac{\mathrm{e}^{-4\sqrt{2}m\lambda\sqrt{|x|}}}{|x|^{\frac{3}{2}}}$$

where $\lambda > 0$, $\lambda^2 = -\varepsilon e \frac{q_0}{m}$ is necessarily positive and $\frac{E}{m} = \varepsilon = \pm 1$.

Remarks.

- The condition $s = 3$ is consistent with earlier differentiability requirements (cf theorem 1)—it ensures that we have 'enough differentiability' of the Dirac field when it is substituted into the Maxwell equations—the electromagnetic potential contains the first derivatives of the Dirac field.
- The rather stronger decay condition $\kappa > 1$ can probably be relaxed to $\kappa > 0$ (or at least $\kappa > \frac{1}{2} - \varepsilon$ see proof of theorem 4) but at the expense of a much more complicated proof. The present condition allows easy use of the multiplication lemma.

Before embarking on a proof of this theorem we will need a couple of preparatory lemmas. In the course of proving the second of these two lemmas, we will also show incidentally that the electric dipole moment must vanish. We are assuming that $|E| = m$ and write $\frac{E}{m} = \varepsilon$. As $\cos\chi \to \varepsilon$, with $|x| \to \infty$ we can take $\chi = n\pi + \zeta$, where $(-1)^n = \varepsilon$ and $\zeta \to 0$, as $|x| \to \infty$.

We require a more careful analysis of equation (12). The imaginary part of this equation is,

$$\begin{aligned}\partial_\alpha \chi + 2(E + eA^0)(n^0 l_\alpha - l^0 n_\alpha) + 2e\sum_{k=1}^{3} A^k(l^k n_\alpha - n^k l_\alpha) + \sqrt{2}m\cos\chi\,(n_\alpha - l_\alpha) \\ - \mathrm{i}\big[\big(\bar{o}_{\dot{A}}\partial^{A\dot{A}}o_A - o_A\partial^{A\dot{A}}\bar{o}_{\dot{A}}\big)n_\alpha + \big(\bar{\iota}^{\dot{A}}\partial_{A\dot{A}}\iota^A - \iota^A\partial_{A\dot{A}}\bar{\iota}^{\dot{A}}\big)l_\alpha \\ + \big(\bar{o}_{\dot{A}}\partial_A{}^{\dot{A}}\iota^A - \iota^A\partial_A{}^{\dot{A}}\bar{o}_{\dot{A}}\big)m_\alpha + \big(\bar{\iota}^{\dot{A}}\partial^A{}_{\dot{A}}o_A - o_A\partial^A{}_{\dot{A}}\bar{\iota}^{\dot{A}}\big)\bar{m}_\alpha\big] = 0.\end{aligned} \tag{17}$$

Assuming the system is asymptotically static and making use of lemma 2 simple calculation reveals,

$$\bar{\iota}^{\dot{A}}\partial_{A\dot{A}}\iota^A + o_A\partial^{A\dot{A}}\bar{o}_{\dot{A}} \in W^{2,2}_{-\kappa-1+2\epsilon}(E_\rho) \qquad \iota^A\partial_A{}^{\dot{A}}\bar{o}_{\dot{A}} - \bar{o}_{\dot{A}}\partial_A{}^{\dot{A}}\iota^A \in W^{2,2}_{-\kappa-1+2\epsilon}(E_\rho).$$

From lemma 2, $n^0 - l^0 \in W^{3,2}_{-\kappa+2\epsilon}(E_\rho)$.

For $\alpha = k = 1, 2, 3$ we make use of $n^k + l^k \in W^{3,2}_{-\kappa}(E_\rho)$ and our previous equation to get

$$2(n^0 l_k - l^0 n_k) - (l^0 + n^0)(l_k - n_k) = (n^0 - l^0)(l_k + n_k) \in W^{3,2}_{-1-\kappa+6\epsilon}(E_\rho).$$

Using lemma 1 and the asymptotic staticity of the system

$$2e\sum_{j=1}^{3} A^j(l^j n_k - n^j l_k) \in W^{2,2}_{-\kappa-1+2\epsilon}(E_\rho).$$

Putting this all together in the equation for $\partial_k \chi$ we have

$$\partial_k \zeta + \sqrt{2}\left[(\varepsilon m + eA^0)\frac{1}{\sqrt{2}}(l^0 + n^0) - \varepsilon m \cos\zeta\right](l_k - n_k) \in W^{2,2}_{-1-\kappa+9\epsilon}(E_\rho).$$

We need to refine our estimate for $(l^0 - n^0)/\sqrt{2}$, noting

$$\left(\frac{l^0 + n^0}{\sqrt{2}}\right)^2 - 1 = \frac{1}{\sqrt{2}}\sum_{j=1}^{3}(l^j + n^j)^2 \in W^{3,2}_{-2\kappa}(E_\rho)$$

and $\left[\frac{1}{\sqrt{2}}(l^0 + n^0) + 1\right]^{-1} \in W^{3,2}_{2\epsilon}(E_\rho)$ we have

$$\frac{1}{\sqrt{2}}(l^0 + n^0) - 1 \in W^{2,2}_{-2\kappa+2\epsilon}(E_\rho).$$

Now use this estimate, together with lemma 1 (to separate the monopole term) and the fact that $(l_k - n_k) + 2l^k \in W^{3,2}_{-\kappa}(E_\rho)$ (remember, $l_k = -l^k$), to get

$$\partial_k \zeta - 2\sqrt{2}\varepsilon m\left[1 - \cos\zeta + \frac{\varepsilon e q_0}{m|x|} + \frac{\varepsilon e}{m}a^0\right]l^k \in W^{2,2}_{-\nu+9\epsilon}(E_\rho)$$

where $\nu = \min[2\kappa, 1+\kappa]$. For convenience we will write this equation in a 3-vector notation, using ∇ to denote the gradient and $l = (l^1, l^2, l^3)$; note that (because of asymptotic staticity) we have for the norm of l, $|l|^2 = l.l = \frac{1}{2}$ plus a term in $W^{s,2}_{-\kappa+\epsilon}$ (see proof of lemma 2). We have,

$$\nabla\zeta - 2\sqrt{2}\varepsilon m\left[1 - \cos\zeta + \frac{\varepsilon e q_0}{m|x|} + \frac{\varepsilon e}{m}a^0\right]l \in W^{2,2}_{-\nu+9\epsilon}(E_\rho). \tag{18}$$

We will also require the asymptotically static version of equation (14), the real part of equation (12). Firstly, from lemma 2 we have that

$$\sqrt{2}m^0 = \sqrt{2}\sigma^0_{A\dot{A}}o^A\bar{\iota} = -o_0\bar{\iota}^{\dot{1}} + o_1\bar{\iota}^{\dot{0}} \in W^{3,2}_{-\kappa+2\epsilon}(E_\rho).$$

We have, after using, $(l_k - n_k) + 2l^k \in W^{3,2}_{-\kappa}(E_\rho)$,

$$\frac{1}{R}\nabla R + 2\sqrt{2}\varepsilon m \sin\zeta\, l \in W^{2,2}_{-\min[1,\kappa]+4\epsilon}(E_\rho). \tag{19}$$

Now, to the first of our two lemmas. In fact, this lemma actually gives us the exponential decay result which we use to improve lemma 1 so that we may obtain the much tighter estimate necessary for theorem 5.

Lemma 3. *The Dirac field (and at least its first and second derivatives) of a stationary, isolated, asymptotically static (with $\kappa > 1$, $s = 3$ and $|E| = m$) Maxwell–Dirac field decays exponentially as $|x| \to \infty$, provided $q_0 \neq 0$. In particular,*

$$R < C\frac{\mathrm{e}^{-2k\sqrt{|x|}}}{|x|} \qquad \textit{for any } k \textit{ such that } 0 < k < 2\sqrt{2}m\lambda$$

here $\lambda^2 = -\frac{\varepsilon e q_0}{m}$ is necessarily positive.

Proof. We first note from theorem 4 that $\sin\zeta \in W^{2,2}_{-\frac{1}{2}+4\epsilon}$ so, for ρ large enough, $\zeta \in W^{2,2}_{-\frac{1}{2}+4\epsilon}(E_\rho)$ and $\nabla\zeta \in W^{1,2}_{-\frac{3}{2}+4\epsilon}(E_\rho)$.

Now, taking the divergence of equation (19) and using (19) again to remove the $|\nabla R|^2$ term,

$$\frac{1}{R}\Delta R - 4m^2\sin^2\zeta \in W^{1,2}_{-\frac{3}{2}+\epsilon}(E_\rho) \qquad \text{for any } \epsilon > 0.$$

From the fact that $\nabla\zeta \in W^{1,2}_{-\frac{3}{2}+4\epsilon}(E_\rho)$ we have from (18),

$$1-\cos\zeta+\frac{\varepsilon e q_0}{m|x|}+\frac{\varepsilon e}{m}a^0 = 2\sin^2\frac{\zeta}{2}+\frac{\varepsilon e q_0}{m|x|}+\frac{\varepsilon e}{m}a^0 \in W^{1,2}_{-\frac{3}{2}+4\epsilon}.$$

Our first observation is that $\varepsilon e q_0 < 0$, since from lemma 1 $a^0 \in W^{5,2}_{-\eta}$ with $\eta > 1$. We write $\lambda^2 = -\frac{\varepsilon e q_0}{m}$, and take $\lambda > 0$. We can also use the last inclusion to estimate the term $\sin^2\zeta$. We have,

$$\sin^2\zeta - \frac{2\lambda^2}{|x|} \in W^{1,2}_{-\min[\frac{3}{2},\eta]+\epsilon}(E_\rho).$$

The second-order elliptic equation for R can now be written as

$$\Delta R - 8m^2\left(\frac{\lambda^2}{|x|}+\alpha\right)R = 0 \qquad \text{where } \alpha \in W^{1,2}_{-\min[\frac{3}{2},\eta]+\epsilon}(E_\rho). \tag{20}$$

We will now use the maximum principle utilising a comparison function

$$v(x) = C\frac{\mathrm{e}^{-2k\sqrt{|x|}}}{|x|} \qquad \text{for which } \Delta v - \left(\frac{k^2}{|x|}-\frac{k}{2|x|^{\frac{3}{2}}}\right)v = 0.$$

Now,

$$\Delta[R-v(x)] - 8m^2\left(\frac{\lambda^2}{|x|}+\alpha\right)[R-v(x)] = \left[(8m^2\lambda^2-k^2)\frac{1}{|x|}+\tilde{\alpha}\right]v(x)$$

where $\tilde{\alpha} \in W^{1,2}_{-\min[\frac{3}{2},\eta]+\epsilon}(E_\rho)$. Consequently, for ρ large enough and for every k such that $0 < k < 2\sqrt{2}m\lambda$ we have,

$$\Delta[R-v(x)] - 8m^2\left(\frac{\lambda^2}{|x|}+\alpha\right)[R-v(x)] > 0.$$

Choosing C such that $[R-v(x)]_{|x|=\rho} \leqslant 0$ we have by the maximum principle that $R-v(x) < 0$ on E_ρ. Completing the proof of our lemma. □

Lemma 4. *For a stationary, isolated asymptotically static Maxwell–Dirac system with $\kappa > 1$, $s = 3$ and $|E| = m$ we have the following estimates when $q_0 \neq 0$.*

$$\zeta - \frac{\varepsilon_1\sqrt{2}\lambda}{\sqrt{|x|}} + \frac{\varepsilon_1}{4m|x|} - \frac{\varepsilon_1(16\lambda^4m^2+9)}{96\sqrt{2}\lambda m^2|x|^{\frac{3}{2}}} \in W^{3,2}_{-2+\epsilon}(E_\rho)$$

$$l = \frac{\varepsilon\varepsilon_1}{\sqrt{2}}(\hat{r}-u) \qquad \textit{with} \quad u \in W^{3,2}_{-\frac{1}{2}+\epsilon}(E_\rho)$$

$$\hat{r}.l - \frac{\varepsilon\varepsilon_1}{\sqrt{2}} \in W^{3,2}_{-1+\epsilon} \qquad \textit{and} \qquad \hat{r}.u \in W^{3,2}_{-1+\epsilon}$$

where $\hat{r}$ is the radial unit vector and $(\varepsilon_1)^2 = 1$.

Proof. In all that follows we are assuming that ρ is large enough that the necessary expansions—e.g., $\sin\zeta - \zeta \in W^{2,2}_{-\frac{3}{2}+\epsilon}(E_\rho)$ when $\sin\zeta \in W^{2,2}_{-\frac{1}{2}+\epsilon}(E_\rho)$—can be made on E_ρ.

We begin with the estimate,

$$\sin^2\zeta - \frac{2\lambda^2}{|x|} \in W^{1,2}_{-\min[\frac{3}{2},\eta]+\epsilon}$$

from the proof of lemma 3. Write,

$$\zeta = \frac{\sqrt{2}\varepsilon_1\lambda}{\sqrt{|x|}} + \zeta_1 \qquad \text{where} \quad \varepsilon_1 = \pm 1$$

and substitute into equation (18)

$$\nabla\zeta_1 - \frac{\varepsilon_1\lambda}{\sqrt{2}|x|^{\frac{3}{2}}}\hat{r} - 2\sqrt{2}\varepsilon m\left[\frac{2\sqrt{2}\varepsilon_1\lambda}{\sqrt{|x|}}\zeta_1 + {\zeta_1}^2\right]l \in W^{2,2}_{-2+\epsilon}(E_\rho).$$

We have kept only terms 'less than order $1/|x|^2$' on the left of the equation. The a^0 from A^0 is of order $1/|x|^2$, since we can now improve the result of lemma 1 using lemma 3—from equation (7) we have $\Delta A^0 = 4\pi e\sqrt{2}(l^0 + n^0)R$ so A^0 must be the sum of an harmonic polynomial (of negative degree) and a term which decays exponentially.

Starting with $\zeta_1 \in W^{2,2}_{-\frac{1}{2}+\epsilon}$ (since ζ is) we have that $\nabla\zeta_1 \in W^{1,2}_{-\frac{3}{2}+\epsilon}$, in the first instance. But then our equation (above) implies that $\zeta_1 \in W^{1,2}_{-1+\epsilon}$ and that,

$$\frac{1}{|x|}\hat{r} + 4\sqrt{2}m\varepsilon\zeta_1 l \in W^{-\frac{3}{2}+\epsilon}(E_\rho).$$

Now write $\zeta_1 = -\frac{\varepsilon\varepsilon_2}{4m|x|} + \zeta_2$ ($\varepsilon_2 = \pm 1$) and repeat the process to find that $\zeta_2 \in W^{0,2}_{-\frac{3}{2}+\epsilon}$. As a consequence we have the following estimates for ζ and l,

$$\zeta - \frac{\sqrt{2}\lambda\varepsilon_1}{\sqrt{|x|}} + \frac{\varepsilon\varepsilon_2}{4m|x|} \in W^{0,2}_{-\frac{3}{2}+\epsilon}(E_\rho)$$

$$l = \frac{\varepsilon_2}{\sqrt{2}}(\hat{r} - u) \qquad \text{with} \quad u \in W^{0,2}_{-\frac{1}{2}+\epsilon}(E_\rho).$$

Next we use the fact that, $l.l - \frac{1}{2} \in W^{3,2}_{-\kappa+\epsilon}$, as the system is asymptotically static. We have,

$$l.l - \tfrac{1}{2} = \tfrac{1}{2}(-2\hat{r}.u + |u|^2) \in W^{3,2}_{-\kappa+\epsilon}.$$

It is a simple matter to show that for $u \in W^{3,2}_{2\epsilon} \cap W^{0,2}_{-\frac{1}{2}+\epsilon}$ we have $u \in W^{3,2}_{-\frac{1}{2}+\epsilon}$: begin with a function f in $W^{3,2}_{2\epsilon} \cap W^{0,2}_{-\frac{1}{2}+\epsilon}$ then integrate $\nabla.(\sigma^{2\delta-1} f\nabla f)$ over E_ρ to show that $\nabla f \in W^{0,2}_{-1-\delta+\epsilon}$ for $0 < \delta \leqslant \frac{1}{4}$, iterate this process to eventually find $\nabla f \in W^{0,2}_{-\frac{3}{2}+\epsilon}$ which together with $f \in W^{0,2}_{-\frac{1}{2}+\epsilon}$ shows that $f \in W^{1,2}_{-\frac{1}{2}+\epsilon}$. Now repeat the process with $\partial_k f \in W^{2,2}_{-1+\epsilon} \cap W^{0,2}_{-\frac{3}{2}+\epsilon}$, and so on to eventually get $f \in W^{3,2}_{-\frac{1}{2}+\epsilon}(E_\rho)$. With $u \in W^{3,2}_{-\frac{1}{2}+\epsilon}$, we can use the multiplication lemma to get $|u|^2 = u.u \in W^{3,2}_{-1+\epsilon}$. Next we use the fact that, $l.l - \frac{1}{2} \in W^{3,2}_{-\kappa+\epsilon}$, as the system is asymptotically static. We have,

$$l.l - \tfrac{1}{2} = \tfrac{1}{2}(-2\hat{r}.u + |u|^2) \in W^{3,2}_{-\kappa+\epsilon}.$$

So that $\hat{r}.u \in W^{3,2}_{-1+\epsilon}$, as $\kappa > 1$.

We also note that an argument similar to that used above shows that as $\zeta_2 \in W^{3,2}_{\epsilon} \cap W^{2,2}_{-\frac{1}{2}+\epsilon} \cap W^{0,2}_{-\frac{3}{2}+\epsilon}$ we must have $\zeta_2 \in W^{3,2}_{-\frac{3}{2}+\epsilon}$. We can now substitute,

$$\zeta_2 = \frac{\alpha_0}{|x|^{\frac{3}{2}}} + \zeta_3 \qquad \text{where} \quad \alpha_0 = \varepsilon_1\frac{(16\lambda^4 m^2 + 9 - 96m^2 d.\hat{r})}{96\sqrt{2}\lambda m^2}$$

here d is a constant vector arising from the expansion of $\frac{\varepsilon e}{m}A^0 = \frac{-\lambda^2}{|x|} + \frac{d.\hat{r}}{|x|^2} + O\left(\frac{1}{|x|^3}\right)$. We find that $\zeta_3 \in W^{3,2}_{-2+\epsilon}$. We note from equation (18) that $\hat{\theta}.\nabla\zeta$ and $\hat{\phi}.\nabla\zeta$ are both in $W^{3,2}_{-2+\epsilon}$—here $\hat{\theta}$ and $\hat{\phi}$ are the angular unit vectors orthogonal to $\hat{r}$. Consequently we must have $d = 0$.

So, as d gives rise to the electric dipole moment, we see that *the electric dipole moment must vanish.*

We still have to show that $\varepsilon_2 = \varepsilon\varepsilon_1$ to obtain the precise statements of the lemma. This is easily done by taking the estimates for $\sin\zeta$ and l and substituting them into equation (19)—to obtain exponential decay (rather than growth!) we require $\varepsilon\varepsilon_1\varepsilon_2 = 1$. □

Proof of theorem 5. Armed with lemma 4 the theorem is remarkably simple to prove. We start with equation (5), which can be written as

$$\frac{1}{R}l.\nabla R + \sqrt{2}\varepsilon m \sin\zeta + \nabla.l = 0.$$

Which gives,

$$\frac{\hat{r}.\nabla R}{R} - \frac{u.\nabla R}{R} + 2\varepsilon_1 m \sin\zeta + \frac{2}{r} - \nabla.u = 0.$$

Using equation (19) we have (as $u.l \in W^{3,2}_{-1+\epsilon}$)

$$\frac{u.\nabla R}{R} \in W^{2,2}_{-\frac{3}{2}+\epsilon}.$$

Noting that $\nabla.u \in W^{2,2}_{-\frac{3}{2}+\epsilon}$, we have, using lemma 4,

$$\frac{\hat{r}.\nabla R}{R} + 2\sqrt{2}\frac{m\lambda}{\sqrt{|x|}} + \frac{3}{2|x|} \in W^{2,2}_{-\frac{3}{2}+\epsilon}(E_\rho).$$

Consequently, we have

$$R = K\frac{\mathrm{e}^{-4\sqrt{2}m\lambda\sqrt{|x|}+\beta}}{|x|^{-\frac{3}{2}}}$$

where $\beta \in W^{3,2}_{-\frac{1}{2}+\epsilon}$ and K may depend on $\frac{x^i}{|x|}$. From equation (19) we see that $\ln K \in W^{3,2}_{\epsilon}$ so that K is a bounded function. The Sobolev inequality implies $|\beta| < C/|x|^{-\frac{1}{2}+\epsilon}$. The result now follows by bounding, Ke^{β}. □

8. Discussion

The results of this paper show that under fairly weak asymptotic conditions the solutions of the stationary Maxwell–Dirac equations are highly localized—the Dirac field decays exponentially, the solutions are 'particle-like'.

It is worth emphasizing again that our results are based purely on rather weak asymptotic regularity and decay assumptions. Nothing is assumed about the behaviour of the fields in the interior region B_ρ. Of course, if we were looking at a complete solution we would need to match the 'interior' to the 'exterior' asymptotic region.

Another important point to note is that all the results require the Maxwell equations only to obtain the decay conditions of lemma 1 and the improved decay required for theorem 5. If this decay is given, *a priori*, then the results apply to the 'Dirac equation in an external field' as it is usually presented.

One question which needs to be addressed is the possible extension of the electric neutrality theorem of [16] to the asymptotically static case. But we leave this to a future paper.

Appendix A. 2-spinors and spinor dyads

We collect here a number of facts relating to 2-spinor dyads and their associated null vectors. We give only a brief statement of the facts, for details the reader should consult the book of Penrose and Rindler [15].

- 2-spinor indices are raised and lowered with ϵ^{AB} and ϵ_{AB} (summation on repeated indices), $\xi^A = \epsilon^{AB}\xi_B$ and $\xi_A = \epsilon_{BA}\xi^B$ for any 2-spinor ξ^A.

- $o_A \iota_B - o_B \iota_A = \epsilon_{AB}$ and $o^A \iota^B - o^B \iota^A = \epsilon^{AB}$, where
$$(\epsilon_{AB}) = (\epsilon^{AB}) = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}.$$
- $o_A \iota^A = -o^A \iota_A = 1$ and $\bar{o}_{\dot{A}} \bar{\iota}^{\dot{A}} = -\bar{o}^{\dot{A}} \bar{\iota}_{\dot{A}} = 1$.
- The van der Waerden symbols $\sigma^{\alpha}_{A\dot{A}}$ connect Minkowski vectors to 2-spinors and vice versa. The $\sqrt{2}(\sigma_k{}^{A\dot{A}})$ (with $k = 1, 2, 3$) are simply the Pauli matrices and $\sqrt{2}(\sigma_0{}^{A\dot{A}})$ is the identity matrix. We have, $\sigma^{A\dot{A}}_{\alpha} \sigma_{\beta A\dot{A}} = \eta_{\alpha\beta}$ the Minkowski metric, and $\sigma_{\alpha A\dot{A}} \sigma^{\alpha}{}_{B\dot{B}} = \epsilon_{AB} \epsilon_{\dot{A}\dot{B}}$.
- Because of these relations the null vectors $l^\alpha = \sigma^{\alpha}_{A\dot{A}} o^A \bar{o}^{\dot{A}}$, $n^\alpha = \sigma^{\alpha}_{A\dot{A}} \iota^A \bar{\iota}^{\dot{A}}$, $m^\alpha = \sigma^{\alpha}_{A\dot{A}} o^A \bar{\iota}^{\dot{A}}$ and $\bar{m}^\alpha = \sigma^{\alpha}_{A\dot{A}} \iota^A \bar{o}^{\dot{A}}$ form a null tetrad; with l_α and n_α real, and m_α complex. We have, $l^\alpha l_\alpha = 0$, $n^\alpha n_\alpha = 0$, $m^\alpha m_\alpha = 0$, $l^\alpha m_\alpha = 0$, $l^\alpha \bar{m}_\alpha = 0$, $n^\alpha m_\alpha = 0$, $n^\alpha \bar{m}_\alpha = 0$, $l^\alpha n_\alpha = 1$ and $m^\alpha \bar{m}_\alpha = -1$.
- For any vector X_α we have,
$$X_\alpha = (n^\beta X_\beta) l_\alpha + (l^\beta X_\beta) n_\alpha - (\bar{m}^\beta X_\beta) m_\alpha - (m^\beta X_\beta) \bar{m}_\alpha.$$
- $o^A \bar{o}^{\dot{A}} \partial_{A\dot{A}} f = l^\alpha \partial_\alpha f$, $\iota^A \bar{\iota}^{\dot{A}} \partial_{A\dot{A}} f = n^\alpha \partial_\alpha f$, and so on.

Appendix B. Explicit forms of the Dirac equations

In this appendix we collect together explicit forms of the Dirac and Klein–Gordon equations for stationary Maxwell–Dirac systems.

In this section we use the notation $\partial_z = \frac{\partial}{\partial z}$ etc, $\partial = \partial_x + \mathrm{i}\partial_y$, and for the electromagnetic potential A^α, $A = A^1 + \mathrm{i}A^2$.

The Dirac bi-spinor is,
$$\psi = \mathrm{e}^{-\mathrm{i}Et} \begin{pmatrix} U_A \\ \bar{V}^{\dot{B}} \end{pmatrix}.$$
The Dirac equations are,
$$\begin{aligned}
&\mathrm{i}m\left(\bar{V}^{\dot{0}} - \frac{E}{m}U_0\right) - \bar{\partial}U_1 - \partial_z U_0 - \mathrm{i}e[(A^0 + A^3)U_0 + \bar{A}U_1] = 0 \\
&\mathrm{i}m\left(\bar{V}^{\dot{1}} - \frac{E}{m}U_1\right) - \partial U_0 + \partial_z U_1 - \mathrm{i}e[AU_0 + (A^0 - A^3)U_1] = 0 \\
&\mathrm{i}m\left(U_1 - \frac{E}{m}\bar{V}^{\dot{1}}\right) + \partial\bar{V}^{\dot{0}} - \partial_z \bar{V}^{\dot{1}} + \mathrm{i}e[-(A^0 + A^3)\bar{V}^{\dot{1}} + A\bar{V}^{\dot{0}}] = 0 \\
&\mathrm{i}m\left(U_0 - \frac{E}{m}\bar{V}^{\dot{0}}\right) + \bar{\partial}\bar{V}^{\dot{1}} + \partial_z \bar{V}^{\dot{0}} + \mathrm{i}e[\bar{A}\bar{V}^{\dot{1}} - (A^0 - A^3)\bar{V}^{\dot{0}}] = 0.
\end{aligned} \tag{B.1}$$

The Klein–Gordon equations are easily derived via differentiation of equations (B.1), we give the results for U only,
$$\begin{aligned}
&\Delta U_0 + 2\mathrm{i}e\sum_{j=1}^{3} A^j \partial_j U_0 + \{(E^2 - m^2) + 2eEA^0 + e^2 A^\alpha A_\alpha \\
&\qquad + \mathrm{i}e[\partial_z(A^0 + A^3) + \bar{\partial}A]\}U_0 + \mathrm{i}e[\partial_z\bar{A} + \bar{\partial}(A^0 - A^3)]U_1 = 0 \\
&\Delta U_1 + 2\mathrm{i}e\sum_{j=1}^{3} A^j \partial_j U_1 + \{(E^2 - m^2) + 2eEA^0 + e^2 A^\alpha A_\alpha \\
&\qquad + \mathrm{i}e[-\partial_z(A^0 - A^3) + \partial\bar{A}]\}U_1 + \mathrm{i}e[-\partial_z A + \partial(A^0 + A^3)]U_0 = 0.
\end{aligned} \tag{B.2}$$

Equation (4) gives the electromagnetic potential in terms of the U and V which may in turn be written terms of R, χ and the o and ι. We give only the result for A^0,

$$\begin{aligned} A^0 = \frac{m}{2e}\left[\mathrm{e}^{-\mathrm{i}\chi}(|o_0|^2+|o_1|^2+|\iota^0|^2+|\iota^1|^2)-\frac{2E}{m}\right] \\ +\frac{\mathrm{i}}{2e}\left[\left(\frac{\bar{\partial}R}{R}+\mathrm{i}\bar{\partial}\chi\right)\iota^0 o_1+\bar{\partial}(\iota^0 o_1)+\left(\frac{\partial R}{R}+\mathrm{i}\partial\chi\right)\iota^1 o_0+\partial(\iota^1 o_0)\right. \\ \left.+\left(\frac{\partial_z R}{R}+\mathrm{i}\partial_z\chi\right)\iota^0 o_0+\partial_z(\iota^0 o_0)-\left(\frac{\partial_z R}{R}+\mathrm{i}\partial_z\chi\right)\iota^1 o_1-\partial_z(\iota^1 o_1)\right]. \end{aligned} \tag{B.3}$$

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